Measure Theory

Dr Ram Asrey Rajput

Bundelkhand College
Jhansi, UP, India-28001

February 2024

Length of interval

Length of an interval is generalized as measure of interval and length of an interval is defined as the difference of end points of interval. Thus if I is one of intervals $[a, b]$, (a, b), $(a, b]$, $[a, b)$ i.e. closed, open, open closed, closed open respectively, then length of interval I is denoted $\mid I \mid$ by defined as $\mid I \mid = b - a$. Futhermore, the empty set and every singleton set of R (set of real number) are zero length interval, i.e., $\mid \phi \mid = 0$ and $\mid \{a\} \mid = 0$ for all $a \in R$.

Length of open set

If G is a open subset of R, then for a countable family of disjoint open intervals $\{I_n\}$, such that $G = \cup_n \{I_n\}$ for all n. Then length of G is defined as

$$|G| = \sum_n |I_n|.$$

Furthermore, if $G_1 \supseteq G_2$, then $|G_1| \geq |G_2|$. Also if $G_1 \cap G_2 = \phi$, then $|G_1 \cup G_2| = |G_1| + |G_2|$.

Length of closed set

If F is a closed subset of an open set G, then length of closed set is defined as

$$|F| = |G| - |G - F|.$$

Outer Lebesgue measure of a set

The outer Lebesgue measure of a set $A \subset R$ is denoted by $m^*(A)$ and defined by

$$m^*(A) = \begin{cases} 0 & \text{if } A = \phi \\ \min \sum_n |I_n| & \text{if } A \neq \phi \end{cases}$$

where $\{I_n\}$ is a countable family of open intervals such that $A \subseteq \cup_n \{I_n\}$. It is also known as Lebesgue exterior measure and also denoted by $m_e(A)$.

Properties of outer Lebesgue measure

1. $m^*(A) \geq 0$ for all subset A of real number.
2. $m^*(A) \leq \sum_n |I_n|$, where $\{I_n\}$ is a countable family of open intervals such that $A \subseteq \cup_n \{I_n\}$.
3. For each $\varepsilon > 0$ (however small), $\exists$ at least one countable family of open intervals $\{I_n\}$ such that $A \subseteq \cup_n \{I_n\}$, then

$$m^*(A) + \varepsilon > \sum_n |I_n|.$$

4. If $A \subseteq B$, then $m^*(A) \geq m^*(B)$

Properties of outer Lebesgue measure

1. If I is the interval of a real line, then $m^*(I) = \mid I \mid$.
2. If $A \subseteq [a, b]$, then $m^*(A) \leq (b - a)$.
3. If $A = \{a\}$, is a singleton set, then $m^*(A) = 0$.
4. For all set A, $m^*(A)$ is unique.
5. If A is an open set of real numbers, then $m^*(A) = |A|$.
6. For each $\varepsilon > 0$ (however small), $\exists$ at least one open set G such that $A \subseteq G$, then
$$m^*(A) + \varepsilon > |G|.$$

Properties of outer Lebesgue measure

1. If $\{A_1, A_2, \ldots\}$ is acountable family of subsets of R, then
$$m^*\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$
2. Set A is countable $\Rightarrow m^*(A) = 0$.
3. A set with outer measure different from zero is uncountable.
4. If A and B are any two disjoint subsets of R, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

Inner Lebesgue measure

It is denoted by $m_*(A)$ or $m_i(A)$ and defined by

$$m_*(A) = b - a - m^*(A^c),$$

where $A \subseteq [a, b]$ ans A^c is the complement of A.

Properties of inner Lebesgue measure

1. $m_*(\phi) = 0$.
2. $m_*(A) \geq 0$ for all $A \subseteq R$.
3. $m_*(A) \geq \mid H \mid$, for any closed set $H \subseteq A$.
4. If H is closed set, then $m_*(H) = \mid H \mid$.
5. For each $\varepsilon > 0$, we can find a closed set $H \subset A$ and satisfy the condition $\mid H \mid > m_*(A) - \varepsilon$.
6. $m^*(A) \geq m_*(A)$ for all A

Lebesgue measurable set

Set A is called Lebesgue measurable if for each set $E \subseteq R$, we have
$m^*(E) = m^*(E \cap A) + m^*(E \cup A^c)$.
Or
The set A is said to be Lebesgue measurable if $m^*(A) = m_*(A)$ and its common value is called $m(A)$.

Theorems

1. Set A of outer measure zero is Lebesgue measurable.
2. Any subset of set A (whose outer measure zero) is also measurable.
3. Every countable set is Lebesgue measurable with measure zero.
4. Let A and B are two sets such that $A \subseteq B$, then $m^*(A) \leq m^*(B)$ and $m_*(A) \leq m_*(B)$.
5. Every bounded open and closed sets are measurable.
6. Set A is measurable $\Longleftrightarrow |G| - |H| < \varepsilon$, where G is a open set containing A and H is closed set contained in A.
7. Union of two measurable sets also measurable.
8. Complement of measurable set is measurable.
9. Intersection of two measurable set is measurable.

Theorems

1. Difference of two measurable set is measurable.
2. Symmetric difference of two measurable set is measurable, i.e. if A and B are measurable then $A \Delta B = (A - B) \cup (B - A)$ is measurable.
3. If A and B are disjoint measurable sets, then $m(A \cup B) = m(A) + m(B)$.
4. Countable union of measurable sets is measurable.
5. Arbitrary union of measurable sets is measurable.
6. First fundamental theorem
   If $E_1, E_2, E_3, \dots$ are pairwise disjoint measurable sets, then $E = E_1 \cup E_2 \cup E_3 \cup \dots$ is measurable and $m(E) = \sum_{i=1}^{\infty} m(E_i)$.
7. Secon fundamental theorem
   If $E_1, E_2, E_3, \dots$ are measurable sets, then $E = E_1 \cap E_2 \cap E_3 \cap \dots$ is measurable.
8. If $< E_1, E_2, E_3, \dots >$ monotonically increasing (or non-decreasing) sequence of measurable sets, then $E = E_1 \cup E_2 \cup E_3 \cup \dots$ is measurable and

$$m(E) = \lim_{n \to \infty} m(E_n).$$

Theorems

1. If $< E_1, E_2, E_3, \ldots >$ monotonically decreasing (or non-increasing) sequence of measurable sets, then $E = E_1 \cap E_2 \cap E_3 \cap \ldots$ is measurable and
$$m(E) = \lim_{n \to \infty} m(E_n).$$
2. Cantor ternary set is measurable and its measure is zero.
3. If E_1 and E_2 are measurable subsets of $[a, b]$, then $m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$.
4. If A, B and C are measurable sets of $[a, b]$, then $m(A) + m(B) + m(C) = m(A \cup B \cup C) + m(A \cap B) + m(B \cap C) + m(C \cap A) - m(A \cap B \cap C)$.
5. A is any measurable set and $A_1, A_2, \ldots, A_n$ are pairwise disjoint measurable sets, then
$$m\left(A \cap \left(\cup_{1 \leq i \leq n} A_i\right)\right) = \sum_{i=1}^{n} m(A \cap A_i).$$
6. For $k > 0$ and $A \subset R$, define $kA = \{x : k^{-1}x \in A\}$, then A is measurable iff kA is measurable and
$$m(kA) = km(A)$$

,

Theorems

1. A family of closed sets whose countable union is closed called F_σ set.
2. A family of open sets whose countable intersection is open called G_δ set.
3. F_σ, G_δ called Borel sets, and every borel sets are Lebesgue measurable.

Boolean ring

A non empty family T of sets which is closed under the formation of unions and differences, i.e., if $A, B \in T$, then $A \cup B \in T$ and $A - B \in T$

Boolean algebra

A non empty family T of sets which is closed under the formation of unions and complements, i.e., if $A, B \in T$, then $A \cup B \in T$ and $A^c \in T$

σ-ring

A non empty family T of sets which is closed under the formation of countable unions and differences.

σ-algebra

A non empty family T of sets which is closed under the formation of countable unions and complements.

Measurable function

A real values function f defined over a measurable set E is said to be Lebesgue measurable if any one of following holds

1. $E(f > a) = \{x \in E : f(x) > a\}$ is measurable for all real number $a \in R$.
2. $E(f \geq a) = \{x \in E : f(x) \geq a\}$ is measurable for all real number $a \in R$.
3. $E(f < a) = \{x \in E : f(x) < a\}$ is measurable for all real number $a \in R$.
4. $E(f \leq a) = \{x \in E : f(x) \leq a\}$ is measurable for all real number $a \in R$.

Characteristic Function

If $A \subset E$ (measurable set), then the function $\phi_A : A \to \{0, 1\}$ is said to be characteristic function of the set A is defined as

$$\phi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Simple function

A real values function $\psi : E \to R$ is said to be simple function if there exist a finite collection $\{A_1, A_2, \ldots, A_n\}$ of disjoint measurable subsets of E, such that $E = \cup_{i=1}^{n} A_i$ and n non-zero real numbers $k_1, k_2, \ldots, k_n$ such that $\psi(x) = k_i$ for all $x \in A_i$, where $i = 1, 2, \ldots, n$. Hence simple function

$$\psi(x) = \sum_{i=1}^{n} k_i \phi_{A_i}(x)$$

Step function

A real values function $s : [a, b] \to R$ is said to be step function if there exist a partition $a = x_0 < x_1 < \cdots < x_n$ of [a,b] and n non-zero real numbers $c_1, c_2, \ldots, c_n$ such that $s(x) = c_i$ for all $x \in (x_{i-1}, x_i]$, where $i = 1, 2, \ldots, n$. Hence step function

$$s(x) = \sum_{i=1}^{n} c_i \phi_{(x_{i-1}, x_i]}(x)$$

Theorem

Every step function is a simple function but converge is not true.

Results

1. A constant function over a measurable set is measurable.
2. f is measurable over a measurable set E , then f is also measurable over a measurable subset A of E.
3. $f(x)$ and $g(x)$ are measurable functions, then $\max\{f(x), g(x)\}$ and $\min\{f(x), g(x)\}$ are measurable.
4. If f is a measurable set on measurable set E, then cf, $-f, f+c, |f|, f^2, \frac{1}{f}$ (if $f \neq 0$) are measurable.
5. If f and g are measurable sets on a measurable set E, then $f+g$, $f-g$, fg and $\frac{f}{g}(g(x) \neq 0)$ are measurable.
6. A continuous function f defined over a measurable set E is measurable, converse is not true.
7.