Topology

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Topological space in terms of open set

Let X be a set and T be the collection of subsets of set X satisfying the conditions as

1. $\phi \in T$ and $X \in T$.

2. If $G_1, G_2, \ldots, G_n \in T$, then $G_1 \cap G_2 \cap \ldots \cap G_n \in T$ i.e. finite intersection belong to T.

3. If $G_i \in T$ for every $i \in \nabla$ where ∇ is a index set, then $\cup_{i \in \nabla} G_i \in T$, i.e. arbitrary union also belong to T.

Then family of subsets T is said to be topology and structure (X, T) is said to be topological space. Moreover every member of T is called T-open set.

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Example

Let $X = \{a, b, c\}$ and let families of sets are as follows
$T_1 = \{\phi, X\}$
$T_2 = \{\phi, \{a\}, \{b, c\}, X\}$
$T_3 = \{\phi, \{a, b\}, \{b, c\}, X\}$
$T_4 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Now families T_1, T_2, T_4 satisfies all three conditions to being topology, whereas T_3 does not satisfy second condition finite intersection property as $\{a, b\} \cap \{b, c\} = \{b\} \notin T$, and hence T_3 is not a topology.

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Classification of topology

Indiscrete topology

The collection of whole set and empty set is called indiscrete topology for example T_1 is indiscrete topology and (X, T_1) is said to be indiscrete topological space.

Discrete topology

If T is the collection of all subsets of X i.e., T is the power set of X, and T satisfies all conditions to be a topology, then T is called discrete topology and (X, T) is called discrete topological space. For example in above example topology T_4 is a discrete topology.

Cofinite topology

If topology T be a collection of all those sobsets of X whose complements are finite sets together with empty set is called cofinite topology or finite compliment topology.

Cocountable topology

If topology T be a collection of all those sobsets of X whose complements are countable sets together with empty set is called cofinite topology or finite compliment topology.

Usual topology

It exist on the set of real numbers R having the collection T of those subsets G of R with the property that, for each $x \in G$, there exist $\varepsilon > 0$, such that $(x - \varepsilon, x + \varepsilon) \subset G$. This topology is said to be usual topology and (R, T) is said to be usual topological space on R. This topology also called standard topology or Euclidean topology.

Lower limit topology

If topology T be a collection of all those sobsets of R defined as follow

1. $\phi \in T$.
2. Subsets $G \in T$ with the property that, for each $x \in G$, there exist a right half open interval $[a, b)$, where $a, b \in R$ and $a < b$ such that $x \in [a, b) \subset G$.

Upper limit topology

If topology T be a collection of all those sobsets of R defined as follow

1. $\phi \in T$.
2. Subsets $G \in T$ with the property that, for each $x \in G$, there exist a right half open interval $(a, b]$, where $a, b \in R$ and $a < b$ such that $x \in (a, b] \subset G$.

Comparison of topologies

Let T_1 and T_2 are two topologies for set X. We say T_1 is coarser or weaker or smaller then T_2 if and only if $T_1 \subset T_2$ i.e. every T_1 open set is T_2 open set. Furthermore in same way topology T_2 is said to be finer or stronger or larger than topology T_1. For example in set X, indiscrete topology is coarsest topology and discrete topology is finest topology. Next if $T_1 \subset T_2$ or $T_2 \subset T_1$ then we say T_1 and $_2$ are comparable otherwise not comparable.

Intersection and union of topology

1. Arbitrary intersection of topologies for X is a topology for X.
2. Arbitrary union is not necessary a topology fot X.

Theorem

Two topologies T_1 and T_2 are called equal if

1. T_1 is finer than T_2
2. T_2 is finer than T_1

Metric topology

Let (X, d) is a metric space, and T is a family of subsets such that

1. $\phi, X \in T$.
2. $G \in T$ having property that for each $x \in G$, there exist $r > 0$, such that the open sphere $S(x, r)$ is contained in G.

Then T is called matric topology.

Metrizable space

If T is a metric topology, then (X, T) is called metrizable space.

Closed set

Let (X, T) is a topological space, a subset F of X is said to be T-closed if and only if its complement F^c is T-open.

Each member of T is called T-open set, whereas two members ϕ and X are both T-open as well as T-closed. Also each member of discreate and indiscreate topology are both T-open as well as T-closed.

Door space

(X, T) is a topological space is called door space iff every subset of X is either open or closed.

Theorem

Arbitrary intersection and finite union of closed sets is closed, but arbitrary union and finite intersection of open sets is open.

Topological space in terms of closed set

Let X be a set and T be the collection of subsets of set X satisfying the conditions as

1. $\phi \in T$ and $X \in T$.
2. If $F_1, F_2, \ldots, F_n \in T$, then $F_1 \cup F_2 \cup \ldots \cup F_n \in T$ i.e. finite union belong to T.
3. If $F_i \in T$ for every $i \in \nabla$ where ∇ is a index set, then $\cap_{i \in \nabla} F_i \in T$, i.e. arbitrary intersection also belong to T.

Then family of subsets T is said to be topology and structure (X, T) is said to be topological space. Moreover every member of T is called T-closed set.

Neighbourhood

(X, T) is a topological space and let $x \in X$. A subset N of X is called T-neighbourhood of point x iff $\exists$ a T-open set G, such that $x \in G \subset N$. Simiarly N is called a T-neighbourhood of $A \subset X$ iff $\exists$ a T-open set G, such that $A \subset G \subset N$.

Theorem

A subset of a topological space is open iff it is neighbourhood of each of its points.

Continuity in matric space

Let (X, d) and (Y, ρ) are matric spaces. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$, iff for given $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon,$$

or

$$f[S(x_0, \delta)] \subset S(f(x_0, \varepsilon)).$$

Continuity in topological space

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$, iff for every T_2-neighbourhood M of $f(x_0)$, there exist T_1-neighbourhood N of x_0, such that

$$f(N) \subseteq M$$

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces and $\{x\}$ is a singleton subset of X. Then a function $f : X \to Y$ is continuous at $x \in X$

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous X iff f is continuous at every point of X.

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be open continuous in X iff the inverse image under f of every open set in Y is open in X.

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Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous at $x \in X$ iff the inverse image under f of every T_2-neighbourhood of $f(x)$ is T_1-neighbourhood of x..

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous iff for every $A \subset X$

$$f(\bar{A}) \subseteq \overline{f(A)}.$$

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous iff for every $B \subset Y$

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B}).$$

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous iff for every $B \subset Y$

$$(f^{-1}(B))^o \supseteq f^{-1}(B^o).$$

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be continuous iff for every $B \subset Y$

$$(f^{-1}(intB)) \subseteq int(f^{-1}(B)).$$

Theorem

Let (X, T_1), (Y, T_2) and (Z, T_3) are topological spaces. functions $f : X \to Y$ and $g : X \to Y$ are continuous, then the composite function $gof : X \to Z$ is continuous.

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Homomorphism

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is said to be homomorphism, if following conditions holds

1. f is bijective or one-one onto.
2. f is $T_1 T_2$ continuous.
3. f^{-1} is $T_2 T_1$ continuous.

Theorem

Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \to Y$ is homomorphism, then f is open as well as closed continuous or bicontinuous.

Topological property

Property of a space which invariant under homomorphism.

Separated set

Let (X, T) is a topological space and $A, B \in T$ are called separated sets if $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$.

Theorem

Let (X, T) is a topological space and $A, B \in T$ are separated sets, then $C \subset A$ and $D \subset B$ will be separated.

Theorem

Let (X, T) is a topological space and two closed (open) subsets $A, B \in T$ are separated sets iff they are disjoint.

Theorem

Let (Y, T_Y) is a topological subspace of topological space (X, T) and let subsets $A, B \in T_Y$, then A, B are T-separated iff they are T_Y-separated.

Connected sets

Let (X, T) is a topological space and $A, B \in T$ are called connected sets if they are not separated.

Theorem

Continuous image of a connected space is connected

Theorem

If f is a continuous of a connected space X into R, then $f(X)$ is a interval

Component

A maximal connected subset of topologicalspace (X, T) is called component.

Theorem

If topological space (X, T) is connected, then X has only one component X itself.

Theorem

If (X, T) is a discreate topological space, then every singleton set is a component.

Theorem

If (X, T) is a topological space, then each cmponent is a closed set.

Theorem

Each point in X contained in exactly one component of X.

Theorem

Each connected subset of X which is both open and closed is a component of X.

Totally Disconnected space

A topological space (X, T) is said to be totally disconnected iff for each pair of distinct points $x, y \in X$, there exist a disconnection $A \cup B$ of X such that $x \in A$ and $y \in B$.

Theorem

Discrete space is totally disconnected.

Local base

Let (X, T) is a topological space, a non-empty collection $B(x)$ of T-nbds of x is called a local base of x iff for every T-nbd of N of x there exist a $B_1 \in B(x)$ such that $B_1 \subseteq N$.

Example

Let $X = \{a, b, c, d, e\}$ and $T = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$. Local base at $x = a$ are

1. $B(a) = \{\{a\}\}$.
2. $B(a) = \{\{a\}, \{a, b\}\}$.
3. $B(a) = \{\{a, b\}\}$ is not a local base

Since for every T-nbd N of a in $\{X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$, there exist at least one $B_1 \in B(a)$ which is contained in every N. In (3) is not a local base, since there exist no $B_1 \in B(a)$ which is contained in $N = \{a\}$ (if we take).

Base

Let (X, T) is a topological space, a non-empty collection $B \subset T$ is called base iff for every point $x \in X$ and every T-nbd N of x there exist a $B_1 \in B$ such that $x \in B_1 \subseteq N$. Or simply if we can form T with the union of members of B, then B is a base.

First countable space

Let (X, T) is a topological space is called firt countable space, If each point of X has a countable local base

Second countable space

Let (X, T) is a topological space is called second countable space, iff there exist a countable base for T

Theorem

Every metric space is first countable.

Theorem

Property of first countable is hereditary property i.e., if (X, T) is first countable space then its all subspaces also be first countable space.

Theorem

Property of first countable is topological property i.e., if (X, T) is first countable space then its homeomorphic image $f(X)$ also be first countable space under a homeomorphism $f : X \rightarrow Y$.

Theorem

Property of secon countable is hereditary property i.e., if (X, T) is secon countable space then its all subspaces also be secon countable space.

Theorem

Property of second countable is topological property i.e., if (X, T) is second countable space then its homeomorphic image $f(X)$ also be second countable space under a homeomorphism $f : X \to Y$.

Lindelof space

A topological space is called Lindelof space iff every T-open cover of X has a countable subcover.

Theorem

Every compact space is Lindelof space but converse is not true.

Theorem

Every second countable space is Lindelof space.

Theorem

Every closed subspace of Lindelof space is Lindelof space.

Separable space

A topological space (X, T) is called separable space iff X contains a countable dence subset, i.e., there exist a countable subset A of X, such that $\overline{A} = X$.

Theorem

Discrete space is separable iff X is countable.

Theorem

Separability is a topological property.

Theorem

Every second countable space is separable.

T_0-space or Kolmogorov space

A topological space is called T_0-space or Kolmogorov space iff for any two distinct points $x, y \in X$ there exist an open set $G \in T$ such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$. For example Discrete space is T_0 space but indiscrete space is not T_0 space.

T_1-space or Frechet's space

A topological space is called T_1-space or Frechet's space iff for any two distinct points $x, y \in X$ there exist two open sets $G, H \in T$ such that $x \in G$ but $y \notin G$ and $x \notin H$ and $y \in H$. For example Discrete space is T_1 space but indiscrete space is not T_1 space.

Theorem

If (X, T) is a topological space, then following are equivalent

1. (X, T) is a T_1 space.
2. Every singleton subset of X is closed.
3. Every finite subset of X is closed.
4. The intersection of all neighbourhoods of an arbitrary point in X is a singleton.

T_2-space or Hausdorff space or separated space

A topological space is called T_2-space or Hausdorff space or separated space iff for any two distinct points $x, y \in X$ there exist two disjoint neighbourhoods $M, N \in T$ such that $M \cap N = \phi$.

Urysohn's Lemma

A topological space (X, T) is normal if and only if for every pair of disjoint nonempty closed subsets $C, D \subseteq X$ there is a continuous function $f : X \to \{0, 1\}$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.